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Single Plane and Multi-Plane Rotor Balancing Using Only Amplitude

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ABSTRACT

Both analytic and graphical means of balancing using only $1\times$ rotor amplitude vibration data with no phase data are developed that allow the trial weights to have the different magnitudes. Traditional methods of amplitude only balancing require the use of trial balance weights of the same size. This new methodology uses *relative influence coefficients* for single plane and multi-plane balancing. An extension to least squares balancing procedures is shown. The matrix based formulation lends itself to computer implementation.

1 NOMENCLATURE

$\vec{\mathbf{u}}_i$	response vector with trial weight i
$\vec{\mathbf{x}}_i$	effect vector for trial weight i
$\vec{\mathbf{u}}$	original response vector
θ	angle that $\vec{\mathbf{x}}_1$ makes with $\vec{\mathbf{u}}$ (See Figure 1.)
α	angle between the first trial weight and the second
β	angle between the first trial weight and the third
x_i	$\lvert\vec{\mathbf{x}}_i\rvert$
r_i	magnitude of the i^{th} trial weight
γ_i	angle of the i^{th} trial weight
ϕ	angle between the original vibration vector and the vector that represents the effect of adding a trial weight at 0° ($\phi = \theta - \gamma_1$, θ in Figure 1.)
u	$\lvert\vec{\mathbf{u}}\rvert$
u_i	$\lvert\vec{\mathbf{u}}_i\rvert$
W_i	magnitude of trial weight i
$\mathbf{h}$	*relative influence coefficient*

2 INTRODUCTION

Sometimes only the amplitude information is available to balance a rotor (no phase information). Such instances include times when the instrumentation is limited to a (single channel) spectrum analyzer or simple filter based amplitude meter; there are times when only the amplitude of the raw un-filtered vibration signal (presuming the signal is mostly $1\times$ vibration) is available for balancing. At times difficulties arise in obtaining an observable section of the shaft for a phase reference when the rotor does not have a dedicated trigger signal. The synchronous trigger can be difficult to record, and the phase from a recording may not be reliable. In such cases one might balance a rotor using only amplitude.

G. B. Karelitz ((7)), in the Research Department of Westinghouse Electric & Manufacturing Company, used a three trial weight method to balance turbine generators. This graphical technique used an *unbalance finder* to locate the mass imbalance; the *unbalance finder* consisted of four transparent strips held together with a pivot at one end. The method could be used with trial weights of unequal magnitudes.

F. Ribary ((8)) presented a graphical construction that balanced using only the amplitude taken from an initial run and three trial weight runs. Somervaille ((9)) considerably simplified the graphical construction of Ribary ((8)). Somervaille's construction is the four circle method of balancing without phase. The four circle method, as it is generally used now, can be found in C. Jackson ((6)).

K. R. Hopkirk ((3, 4)) derived an analytical method using only the amplitude information from the response to solve for the required balance weights. Hopkirk's method required three trial weights; he, as other authors have, assumed that the trial weights would all have the same magnitude. His solution produces the vector that represents the effect of the initial trial weight referenced to the original vibration whose angle remains unknown. These papers (3, 4) presented a two plane balance solution using only the amplitude of the vibration — no vibration phase was required for a two point exact-point balance.

More recently, L. E. Barrett, D. F. Li, and E. J. Gunter ((1)) adapted the technique to balance a rotor through two modes using modal balance weights; E. J. Gunter, H. Springer, and R. R. Humphris ((2)) used modal balancing without phase to balance a rotor through three modes. In 1987 L. J. Everett presented a two plane balance procedure that did not use the vibration phase.

However, all the methods other than (7) required the use of identical trial weights in the balancing procedures. This paper will eliminate this requirement and give both graphical and analytical solutions to the basic problem of finding *relative influence coefficients*.

3 FUNDAMENTAL EQUATIONS

This section gives the theoretical basis for *relative influence coefficients* used in balancing using only amplitude. Similar to the derivation of K. R. Hopkirk ((3, 4)), one writes the Equations (1) relating the vibration vectors with each of three trial weights to the original vibration and the effect of adding each trial weight. These equations use the law of cosines with an exterior angle (hence the term of the form $+2ab\cos(\gamma)$ instead of $-2ab\cos(\gamma)$). The angles and the amplitude of the trial weight effects are unknowns. Solving for these unknowns will produce a type of influence coefficient that can be used for balancing. Each of the trial weights may have different magnitudes in the following derivations. Figure 1 illustrates the geometry of the vibration vectors for adding three trial weights at one location.

$$|\vec{\mathbf{u}}_1|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{x}}_1|^2 + 2|\vec{\mathbf{u}}||\vec{\mathbf{x}}_1|\cos(\phi+\gamma_1) \quad (1a)$$

$$|\vec{\mathbf{u}}_2|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{x}}_2|^2 + 2|\vec{\mathbf{u}}||\vec{\mathbf{x}}_2|\cos(\phi+\gamma_2) \quad (1b)$$

$$|\vec{\mathbf{u}}_3|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{x}}_3|^2 + 2|\vec{\mathbf{u}}||\vec{\mathbf{x}}_3|\cos(\phi+\gamma_3) \quad (1c)$$

The angles are measured either in the direction of rotation or against the direction of rotation, but one convention must be adhered to throughout. Angles measured against rotation are called *lag* angles; while angles measured with rotation are called *lead* angles.

3.1 Solving the Equations

One should first note that $x_i = r_i|\mathbf{h}|$, where $\mathbf{h}$ is a complex valued relative influence coefficient. Then expanding the trigonometric terms using the cosine additon formula, re-arranging terms, and making the substitutions $A = |\mathbf{h}|\cos(\phi)$, $B = |\mathbf{h}|\sin(\phi)$, and $C = |\mathbf{h}|^2$ in the Equations (1) results in equations written in matrix form as (2).

$$\begin{bmatrix} 2ur_1\cos(\gamma_1) & -2ur_1\sin(\gamma_1) & r_1^2 \\ 2ur_2\cos(\gamma_2) & -2ur_2\sin(\gamma_2) & r_2^2 \\ 2ur_3\cos(\gamma_3) & -2ur_3\sin(\gamma_3) & r_3^2 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} u_1^2 - u^2 \\ u_2^2 - u^2 \\ u_3^2 - u^2 \end{pmatrix} \quad (2)$$

A numerical solution to the above equation can be easily implemented, and A, B, and C found. The effect of adding the first trial balance weight relative to the original vibration can be written in

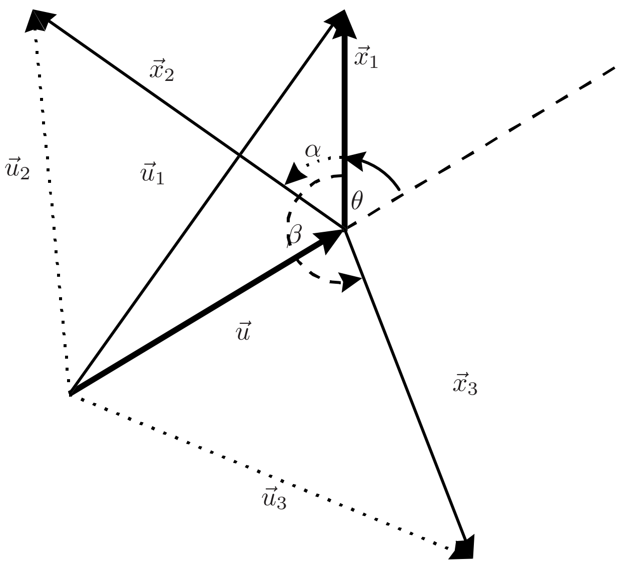


Figure 1: $1\times$ Vibration Vectors with Three Trial Weights In a Single Plane

rectangular coordinates as a vector or as a complex coefficient as required. Equation (3) shows the complex form that is used in computations.

$$\overbrace{\mathbf{h} = A + iB}^{\textit{Complex Coefficient}} \tag{3}$$

This scaled coefficient represents the effect relative to the original vibration angle of adding a trial weight of unit magnitude at $0°$. Thus $\mathbf{h}$ is an influence coefficient relative to the unknown original vibration angle which will be called a *relative influence coefficient*.

3.2 Using More Than Three Trial Weights

One of the advantages often touted for balancing using only amplitude is its inherent averaging. This averaging can be enhanced by using more than three trial weights. can be added considering the first trial weight. In the presence of noise or non-linearities an Equation (4) similar to Equation (2) but can be written but for k trial weights. It includes the resulting errors, e_j.

$$\begin{bmatrix} 2ur_1\cos(\gamma_1) & -2ur_1\sin(\gamma_1) & r_1^2 \\ 2ur_2\cos(\gamma_2) & -2ur_2\sin(\gamma_2) & r_2^2 \\ \vdots & \vdots & \vdots \\ 2ur_k\cos(\gamma_k) & -2ur_k\sin(\gamma_k) & r_k^2 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} u_1^2 - u^2 \\ u_2^2 - u^2 \\ \vdots \\ u_k^2 - u^2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_k \end{pmatrix} \tag{4}$$

This can be solved for the best fit that minimizes the sum of the squares of the errors (a least squares fit) by using the Moore-Penrose generalized inverse (See Horn and Johnson ((5)).) denoted by '$^+$'. The solution is

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{bmatrix} 2ur_1\cos(\gamma_1) & -2ur_1\sin(\gamma_1) & r_1^2 \\ 2ur_2\cos(\gamma_2) & -2ur_2\sin(\gamma_2) & r_2^2 \\ \vdots & \vdots & \vdots \\ 2ur_k\cos(\gamma_k) & -2ur_k\sin(\gamma_k) & r_k^2 \end{bmatrix}^+ \begin{pmatrix} u_1^2 - u^2 \\ u_2^2 - u^2 \\ \vdots \\ u_k^2 - u^2 \end{pmatrix} \tag{5}$$

The *relative influence coefficient*, $\mathbf{h}$, is defined as before by Equation (3).

3.3 Analytical Expression for Solution

One can give an analytical expression for the solution to the equation represented in (2). To find the effects vector and hence the relative influence coefficients only the values of A and B are required. The value of C can be approximated by $A^2 + B^2$. This will be exactly equal to C when the problem is perfectly linear; this occurs for the graphical solution when all three circles intersect in a single point (See Section 4.2.). The analytic solution for A and B is given below.

$$A = \frac{E}{2G} \qquad\qquad B = \frac{F}{2G} \tag{6}$$

The following definitions are used in Equation (6).

$$\begin{aligned}
E &= (r_1 r_3^2 u_2^2 + r_1 u^2 r_2^2 - r_1 r_3^2 u^2 - r_1 u_3^2 r_2^2)\sin(\gamma_1) \\
&\quad + (-r_1^2 r_2 u^2 + r_1^2 r_2 u_3^2 - r_2 r_3^2 u_1^2 + r_2 r_3^2 u^2)\sin(\gamma_2) \\
&\quad + (r_1^2 r_3 u^2 - r_2^2 r_3 u^2 - r_1^2 r_3 u_2^2 + r_2^2 r_3 u_1^2)\sin(\gamma_3) \\
F &= (-u^2 r_1 r_2^2 + u^2 r_1 r_3^2 + r_1 r_2^2 u_3^2 - r_1 r_3^2 u_2^2)\cos(\gamma_1) \\
&\quad + (u^2 r_1^2 r_2 - u^2 r_2 r_3^2 - r_1^2 r_2 u_3^2 + r_2 r_3^2 u_1^2)\cos(\gamma_2) \\
&\quad + (-u^2 r_1^2 r_3 + u^2 r_2^2 r_3 + r_1^2 r_3 u_2^2 - r_2^2 r_3 u_1^2)\cos(\gamma_3) \\
G &= u r_1 r_2 r_3 \big(r_3 \sin(\gamma_1 - \gamma_2) + r_1 \sin(\gamma_2 - \gamma_3) + r_2 \sin(\gamma_3 - \gamma_1)\big)
\end{aligned}$$

The *relative influence coefficient* (relative to the first trial weight), $\mathbf{h}$, is given by Equation (3). This completes the derivation of the theory of *relative influence coefficients*.

4 BALANCING

This section discusses the use of *relative influence coefficients* in balancing. First the usual single plane balance situation will be described. This balance is an exact point balance which means that the number of measurements used in the balance equals the number of balance planes; in this case there is only one measurement and one balance plane.

4.1 Single Plane Balancing Using Influence Coefficients

Single plane exact point balancing can be done using the *relative influence coefficient* vector to cancel the original vibration. To find the corrective imbalance, $\mathbf{W_c}$, one treats all quantities as complex coefficients. The solution to the single plane balance is then given by:

$$\text{Correction Weight} = -\frac{\text{Initial Vibration}}{\text{Influence Coefficient}}$$

In the present context this solution can be written as:

$$\text{Corrective Weight} = \mathbf{W}_c = -\frac{|\mathbf{u}|}{\mathbf{h}} \tag{7}$$

Note that one only knows the amplitude of the original vibration in Equation (7), and $\mathbf{h}$ has its angular reference relative to that of the original vibration which is unknown. Also note that the correction weight in the single plane balance, $\mathbf{W}_c$ will have angle equal to $180 - \angle\mathbf{h}$ in degrees, because the solution is of the form $-real/\mathbf{h}$ where $\mathbf{h}$ is *complex* valued.

4.1.1 Single Plane Balance Example

L. E. Barrett, D. F. Li, and E. J. Gunter ((1)) gave the following example of an experiment as reported in Table 1. This set of measurements resulted from a second mode balance weight combination, two weights 180° out of phase at each end of the rotor. The reported phase angle comes from the location of the balance weight on Disc 1. Each trial weight had a magnitude of 0.17gm per disc.

Table 1: Vibration Amplitudes (1)

Run	Trial Weight Phase	Amplitude μm (mils)
0	Initial Run	$231(9.1)$
1	$0°$	$295(11.6)$
2	$135°$	$160(6.3)$
3	$270°$	$284(11.2)$

Using the methods described here, the modal *relative influence coefficient* is computed using Equation (2). Since the three trial weights have the same magnitude (2) simplifies to the following equation.

$$\begin{bmatrix} 2(9.1)(.17) & 0 & .17^2 \\ 2(9.1)(.17)\cos(135°) & -2(9.1)(.17)\sin(135°) & .17^2 \\ 2(9.1)(.17)\cos(270°) & -2(9.1)(.17)\sin(270°) & .17^2 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 11.6^2 - 9.1^2 \\ 6.3^2 - 9.1^2 \\ 11.2^2 - 9.1^2 \end{pmatrix}$$

Solving the above equation gives the values for A, B, and C.

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 13.5642 \\ 10.6166 \\ 338.4904 \end{pmatrix}$$

One can see that $A^2 + B^2$ does not equal C as it would in a perfectly linear system without noise. For any experimental data like this exact equality will not hold.

One can not determine the actual modal influence coefficient, but one can determine the effect a balance weight has relative to the initial vibration angle.This relative effect vector divided by the magnitude of the first trial weight acts as an influence coefficient relative to the unknown original vibration angle and relative to the first trial weight. This *relative influence coefficient*, **h**, is given below (See Equation (3).).

$$\mathbf{h} = (17.225, 38.0°)$$

The corrective balance weight (relative to the first disc) can be computed using complex arithmetic.

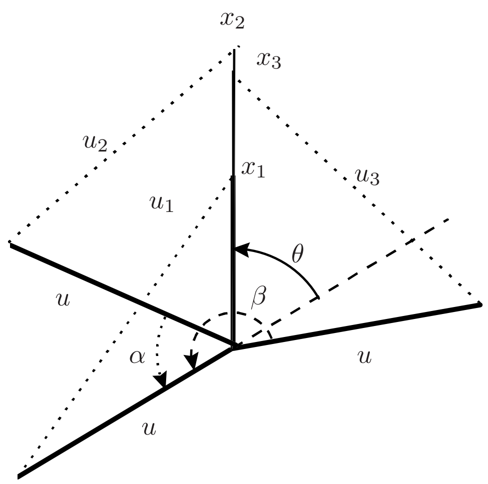
$$\mathbf{c} = \frac{-\text{Original Vibration}}{\text{Relative Influence Coefficient}} = \frac{-9.1}{(17.225, 38.0°)} = (0.5283, 142.0°)$$

The authors of the report (1) used a graphical method and reported a solution of 0.53gm at $140°$ for the corrective balance weight on disc 1; disc 2 would have an equal amount of weight placed $180°$ from that on disc 1. These two solutions differ by about 3.5%.

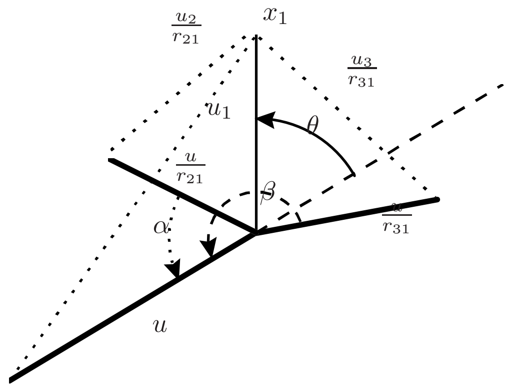
4.2 Single Plane Balancing — Graphical Method

To obtain a graphical solution for a single plane exact point balance, one can rearrange the triangles from Figure 1 to obtain Figure 2(a). In 1954 Somervaille presented a graphical solution for a single plane exact point balance using equal trial weights; in this case he had $|\vec{x}_1| = |\vec{x}_2| = |\vec{x}_3|$. Somervaille used a similar method to arrive at the usual graphical four run method. In order to proceed with a graphical balancing procedure for unequal trial weights, one must first re-scale two of the triangles in Figure 2(a) as has been done in Figure 2(b).

From the figure one can see that three circles would meet at the point determining the effects vector, $|\vec{x}_1|$. The graphical procedure can be stated in the following graphical algorithm.



(a) Unscaled Geometry



(b) Scaled Geometry

Figure 2: $1\times$ Vibration Vectors with Three Trial Weights

1. Record the initial vibration with no trial weights attached, u in Figure 2(b).
2. Determine the first trial weight, $\vec{\mathbf{W}}_1$, with angle γ_1. Run the rotor to get the amplitude u_1.
3. Plot the point (u, γ_1). Draw a circle of radius u_1 using this point as the center.
4. Determine the second trial weight, $\vec{\mathbf{W}}_2$, with angle γ_2. Remove the first trial weight and apply this weight. Run the rotor to get the amplitude u_2.
5. Calculate $r_{21} = W_2/W_1$.
6. Plot the point $(u/r_{21}, \gamma_2)$. Draw a circle of radius u_2/r_{21} using this point as the center.
7. Determine the third trial weight, $\vec{\mathbf{W}}_3$, with angle γ_3. Remove the second trial weight and apply this weight. Run the rotor to get the amplitude u_3.
8. Calculate $r_{31} = W_3/W_1$.
9. Plot the point $(u/r_{31}, \gamma_3)$. Draw a circle of radius u_3/r_{31} using this point as the center.
10. Measure the point of approximate intersection for the three circles, $\vec{\mathbf{x}}_1$.
11. **Solution**

    (a) *Magnitude of Correction Weight*is $u/|\vec{\mathbf{x}}_1|$
    (b) *Angle of Correction Weight* is $180 - \angle\vec{\mathbf{x}}_1$

Note: As before there is no constraint on the angles of the trial weights. However, if all three trial weights are placed at the same angular location , including the location $\pm 180°$, one can not resolve the angle of the correction weight uniquely. There will be two solutions. Using at least two different angles for the trial weights will yield a solution provided one does not use the identical trial weights (amplitude and angle) twice. This holds for both the graphical and analytical methods.

If only one angle were to be used one of can see from Equation (2) that the first and second columns would be linearly dependent. This implies that one of the equations is redundant which means that one does not have enough information to solve the problem.

4.3 Least Squares Balancing

The basic principle used to define the influence coefficient (3) easily extends to multiple measurement points, speeds, and loads. This is accomplished by indexing the measurements by i and trial weights by j to give the $\mathbf{h}_{i,j}$. In the above $\mathbf{h}_{i,j}$ is computed by adding three trial weights (or more) and applying Equation (3) with each measurement location (and speed) i and for each balance plane j. This requires $i \times j$ solutions of the matrix equation (2).

4.3.1 Example The following influence coefficients were obtained from a balance of a 2983 kW (4000 Hp) two pole induction motor. The balance took place on a test stand under no load. The influence coefficients are given in units of μm/gram with an implied balance weight radius equal to the fixed balance plane radius at the balance plane located on the inboard cooling fan which is internal to the motor.

Table 2: Influence Coefficients for Two Pole Induction Motor

	Inboard Fan	
Response Plane	Amplitude μm/gram	Phase degrees
Inboard x	2.261	$112°$
Inboard y	0.889	$71°$
Outboard x	0.203	$85°$
Outboard y	0.203	$87°$

The influence coefficients in Table 2 will be used to simulate the unbalance response for this motor and its balancing. First a simulated $1\times$ vibration at the bearings is given in Table 3. Notice that the outboard end has much lower vibration than the inboard end.

Table 3: Simulated Vibration for Induction Motor

	Vibration	
Response Plane	Amplitude μm	Phase degrees
Inboard x	112.4	60.4
Inboard y	42.5	22.0
Outboard x	9.2	32.1
Outboard y	10.3	32.0

The three trial weights given in order of use are 20g at $0°$, 40g at $105°$, and 50g at $210°$. The amplitudes and phases (reported but not used) of the simulated vibration with each of these trial weights are given in Table 4.

Table 4: Simulated Vibration with Trial Weights

	Trial Weight 1		Trial Weight 2		Trial Weight 3	
Response Plane	Amplitude μm	Phase degrees	Amplitude μm	Phase degrees	Amplitude μm	Phase degrees
Inboard x	144.9	74.6	46.4	111.1	147.3	11.0
Inboard y	55.8	35.9	18.8	77.8	55.4	330.0
Outboard x	12.1	47.6	3.5	92.7	12.9	340.5
Outboard y	13.1	46.7	3.9	78.1	13.8	345.0

The *relative influence coefficient* for the inboard x measurement can be calculated using the first row of Table 4. This computation (to the accuracy of the data from the tables) gives 2.2615g-μm at $51.6°$ for the *relative influence coefficient*, $h_{1,1}$. One can see that $h_{1,1}$ has the same magnitude as the influence coefficients from Table 2; however the angle of $h_{1,1}$ differs from that of Table 2 by $60.4°$, the (unknown) angle of the initial vibration. The *relative influence coefficients* for all the locations are displayed in Table 5; this table was computed using only the amplitudes (with the full accuracy to compare to the original influence coefficients) of the vibration by Equation (5) and the definition for h in Equation (3).

Table 5: Influence Coefficients for Electric Motor

	Inboard Fan	
Response Plane	Amplitude μm/gram	Phase degrees
Inboard x	2.261	51.6°
Inboard y	0.889	49.0°
Outboard x	0.203	52.9°
Outboard y	0.203	55.0°

The least squares balance (un-weighted) using only the original amplitudes of vibration (with $0°$ phase) in the balance computation yields a correction weight of 49.4g at $128.7°$. Using the full influence coefficients from Table 2 with angular information to balance the original vibration with its phase yields the same result. The predicted residual vibration with this correction weight for a linear system would have a maximum amplitude of 2.2μm. Because this is not an exact point balance, one can not balance all measurement points to zero vibration simultaneously.

5 CONCLUSIONS

The technique of balancing using only the amplitudes of vibration has been extended in several ways. Most importantly, this method removes the restriction that the trial weights all have the same magnitude. Solution algorithms have been in matrix form suitable for computer implementation. A graphical method that allows one to use different size trial weights has be described.

The additions to balancing without phase include the following.

Relative influence coefficients have been defined which allow one more flexibility in balancing. Using these *relative influence coefficients* one can perform multi-speed least squares balancing or any other form of influence coefficient balancing with the exception of trim balancing.

A matrix formulation for the generation of the *relative influence coefficients* was developed using three trial weights whose magnitudes may differ from each other.

A matrix formulation that enables the generation of *relative influence coefficients* by applying a least squares fit to data involving more than three trial weights. Again the magnitude of the trial weights may differ.
A graphical procedure to compute the *relative influence coefficients* was described which allows the balancer to use trial weights of differing magnitudes.

The use of trial weights of different magnitudes when balancing without phase has been developed. Balancing without phase has been extended to include multiple speeds (or conditions such as loads). An analytical computation of the *relative influence coefficients* was given.

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